for neutral solution of the equations of stability in the form

$$L = 0 \qquad (11)$$

where

$$L = (\gamma_1 - p + \gamma_2\zeta_k^2)\,[\gamma_3 - p - 2\gamma_4\zeta - {}^1/_4\,(\zeta - \zeta_0)\,\theta^{-2} + 8\gamma_2\zeta^2 + {}^1/_{64}\,\gamma_5\,(\zeta_k^2 + \zeta_k\zeta_{0k} - \zeta_{0k}^2)] -$$
$$- {}^1/_8\,[{}^1/_4\,\zeta_k\theta^{-2} + \gamma_4\,(2\zeta_k - \zeta_{0k}) - 8\gamma_2\,(\zeta\zeta_k + \zeta_0\zeta_k - 2\zeta_0\zeta_{0k} + \zeta\zeta_{0k})]^2$$

The critical value of p is determined from condition (11). If however the condition $L = 0$ is not satisfied, then, as before, p is determined from the condition of minimum of the value L.

Results of calculations of critical value p according to condition (11) are presented in Fig. 5. Each curve in Fig. 5 consists of two parts: the first (before the corner point) is determined by the condition $L = 0$, the second by the condition $L_{\min}$. Comparing results of calculations according to conditions $M = 0$ (Fig. 4) and $L = 0$ (Fig. 5), it is possible to draw the conclusion that the linearization of equations of the precritical state gives acceptable results only in a relatively small region of small values of deflections ζ_{0k}. Naturally, for $\zeta_{0k} = 0$ the results of calculations coincide completely because in this case the starting equations are the same.

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LINEAR PROBLEM OF ROTATIONAL OSCILLATIONS OF AN ELASTICALLY COUPLED RIGID SPHERE IN A VISCOUS FLUID, BOUNDED BY A CONCENTRIC STATIONARY SPHERE

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The rotation of a rigid sphere around its diameter with small angular deflection from stationary position is examined under the influence of an elastic force couple in a viscous medium bounded from the outside by a concentric stationary sphere.

The spectrum of oscillations is investigated in detail. The spectral distributions of angular velocity of the sphere are obtained for any positive value of parameters of the problem. In this connection a qualitative analogy is established between the motion of the sphere and a plane oscillating between parallel walls.

In connection with viscosity measurements of gases Maxwell carried out a mathematical analysis of small rotational oscillations for a rigid flat disk suspended by an elastic thread in a viscous fluid which is confined between parallel stationary planes. Maxwell assumed that the disk executes harmonically damped oscillations. He derived the characteristic equations for the oscillation of the disk and obtained approximate equations for calculation of viscosity for the case where the complex root of the characteristic equation is given from an experiment [1].

With the same purpose Verschaffelt examined the problem of small rotational oscillations of an elastically coupled rigid sphere in a viscous fluid bounded by a concentric stationary sphere [2]. He applied the obtained results to viscosity measurements of dilute gases.

In view of the theoretical and practical interest of problems partially examined in [1] and [2], it was desirable to formulate and solve these problems with consideration of initial conditions without assuming in advance the angular velocity of the rigid disk or sphere to be exponential with a complex index proportional to time. It was also desirable to investigate in detail the characteristic equations for all admissible values of parameters and to give a spectral distribution of solutions. In this framework the problem of longitudinal translational oscillations of an elastically coupled rigid plane in a viscous fluid (mathematically this is identical to the linear problem of rotational oscillations of an infinite flat disk) was studied in paper [3]. Some results of this work are utilized below in the investigation of the spectrum of oscillations of a sphere in the problem of Verschaffelt.

1. Formulation of the problem. A rigid sphere with a radius R_* is suspended by an elastic thread of rotational stiffness M_* and executes small rotational oscillations in the homogeneous fluid with a viscosity η_* and density μ_*.

The fluid is bounded by a concentric and, with respect to the rigid sphere, stationary sphere of radius $R_*' > R_*$. On the surface of the rigid sphere and also on the external boundary the condition of adhesion is satisfied. The moment of inertia of the sphere is equal to K_*. At the initial instant the sphere and the fluid are at rest. The sphere in this case is twisted with respect to the equilibrium position by an angle A_0.

Subsequently the fluid is perturbed into motion only by the sphere. It rotates in undeforming spheres (the angle A_0 is so small that the convective terms in the acceleration of the fluid are insignificant in comparison with the local term). The desired angular velocity ω_* of these spheres depends on time t_* and the radius r_*, $R_*' \geqslant r_* \geqslant R_*$. The angular velocity of the sphere $\omega_{0*}(t_*) = \omega_*(t_*, R_*)$.

The asterisk denotes quantities of nonzero dimension. The parameters A_0, R_*, R_*', K_*, M_*, η_* and μ_* are positive. M_* can be replaced by parameter $k_{0*} = \sqrt{M_* / K_*}$.

Let us introduce the following nondimensional quantities:

$$t = k_{0*} t_*, \quad r = \frac{r_*}{R_*}, \quad r_e = \frac{R_*'}{R_*}, \quad \omega = \frac{\omega_*}{k_{0*}}, \quad \omega_0 = \frac{\omega_{0*}}{k_{0*}}$$

$$\eta = \frac{8\pi R_*^3}{3K_* k_{0*}} \eta_*, \quad \mu = \frac{8\pi R_*^5}{3K_*} \mu_*, \quad \nu = \frac{\eta}{\mu}$$

The solution of the problem will be understood to be a function $\omega(t, r)$ satisfying the following conditions.

1) Function $\omega(t, r)$ is continuous in $(t \geqslant 0,\ r_e \geqslant r \geqslant 1)$ and becomes zero for $t = 0$, $r_e \geqslant r \geqslant 1$ and $t \geqslant 0,\ r = r_e$.

2) In $(t > 0,\ r_e \geqslant r \geqslant 1)$ continuous derivatives $\omega_t, \omega_r, \omega_{rr}$ exist and the following equation is satisfied

$$\omega_t = \nu(\omega_{rr} + 4r^{-1}\omega_r) \tag{1.1}$$

3) The quantity $\omega(t, 1) \equiv \omega_0(t)$ for $t > 0$ satisfies the following equation:

$$\omega_0'(t) + A_0 + \int_0^t \omega_0(\tau)\,d\tau - \eta\omega_r(t, 1) = 0 \tag{1.2}$$

The uniqueness of such a function follows from energetic considerations.

2. Integral representation of the solution. The solution is represented by a Laplace-Mellin integral

$$\omega(t, r) = \frac{1}{2\pi i}\int_{\gamma-i\infty,\ \gamma\geqslant 0}^{\gamma+i\infty} -\frac{A_0}{r^{3/2}}\frac{D(z, r)\,e^{zt}}{D(z, 1)\,\varphi(z)}\,dz \tag{2.1}$$

$$D(z, r) = I_{3/2}(br)\,K_{3/2}(br_e) - I_{3/2}(br_e)\,K_{3/2}(br), \qquad b = \sqrt{z/\nu}, \quad |\arg b| \leqslant {}^1/_2\pi$$

$$\varphi(z) = z^2 + 1 + \eta z\left[\frac{3}{2} - \frac{bD_1}{D(z, 1)}\right]$$

$$D_1 = I_{3/2}'(b)\,K_{3/2}(br_e) - I_{3/2}(br_e)\,K_{3/2}'(b)$$

It is easy to check that $\omega(t, r)$ is continuous together with ω_r in $(t \geqslant 0,\ r_e \geqslant r \geqslant 1)$ and will be an analytical function of t and r in $(t > 0,\ r_e \geqslant r \geqslant 1)$. The derivative ω_t is continuous as a function of t for any r, $r_e \geqslant r \geqslant 1$, but suffers a discontinuity as a function of r at the point $t = 0$, $r = 1$, because

$$\omega_t(0, 1) = -A_0 < 0, \qquad \omega_t(0, 1 + 0) = 0$$

3. Investigation of the spectrum. The expressions $\omega = \operatorname{Re}[e^{zt}u(z, r)]$ for some $u(z, r)$ satisfy all conditions of the formulated problem with the exception of the condition of $\omega(t, r)$ becoming zero for $t = 0$, then and only then, when $z = k$, where k is any root of the function $\varphi(z)$. In this sense the roots $\varphi(z)$ will be points of the spectrum of the problem (completely discrete). We shall elucidate how they are distributed in the plane z.

We introduce the parameters $\lambda = \sqrt{\nu}$, $\varkappa = \eta / \lambda$, $\xi = (r_e - 1)/\lambda$. We fix $\varkappa > 0$, $\xi > 0$, varying the parameter λ in the interval $[0, \lambda_0]$, where λ_0 is an arbitrarily fixed positive number. Let us represent $\Phi(z, \lambda) \equiv \varphi(z)$ through a ratio of singlevalued entire functions of z, which can be expanded in powers of z in series with real coefficients. These functions are continuous as functions of two variables z and λ

$$\Phi(z, \lambda) = \Phi_2(z, \lambda) : \Phi_1(z, \lambda), \qquad \Phi_1(z, \lambda) = \mu\,[I_{3/2}(a)\,K_{3/2}(b) - I_{3/2}(b)\,K_{3/2}(a)]$$

$$\Phi_2(z, \lambda) = (z^2 + 1 + {}^3/_2\lambda\varkappa z)\,\Phi_1 + \mu\varkappa z\sqrt{z}\,[I_{3/2}'(b)\,K_{3/2}(a) - I_{3/2}(a)\,K_{3/2}'(b)]$$

$$\mu = \sqrt{1 + \xi\lambda}\,/\,\xi\lambda, \qquad a = br_e = (1/\lambda + \xi)\sqrt{z}, \qquad b = \lambda^{-1}\sqrt{z}$$

$$\Phi_2(0, \lambda) = \Phi_1(0, \lambda) = 1 + 1/(3\mu^2) > 0$$

We note that Φ_1 and Φ_2 for any $\lambda > 0$ do not have common roots in the plane z. The latter follows, e.g. from equality

$$I_{3/2}(b)\,K_{3/2}'(b) - I_{3/2}'(b)\,K_{3/2}'(b) = -1/b \neq 0$$

For $\lambda = 0$

$$\Phi_1(z, 0) = \frac{\operatorname{sh}(\xi\sqrt{z})}{\xi\sqrt{z}}, \quad \Phi_2(z, 0) = (z^2+1)\,\Phi_1(z, 0) + \frac{\varkappa}{\xi}\, z \operatorname{ch}(\xi\sqrt{z})$$

It was proved [3] that roots $\Phi_2(z, 0)$ are all included in a denumerable set of simple negative roots $k_1 > k_2 > \ldots$ and two roots k_{01}, k_{02}, which are negative (in this case it is possible that $k_{01} = k_{02}$) or complex conjugate with a negative real part. The roots $\zeta_n = -\pi^2\xi^{-2}n^2$, $n = 1, 2, \ldots$, of the function $\Phi_1(z, 0)$ are separated by roots $\Phi_2(z, 0)$

$$0 > \zeta_1 > k_1 > \zeta_2 > k_2 > \ldots$$

In the plane z let a sequence of circumferences Γ_m be constructed, $m = 1, 2, \ldots$, with the center $z = 0$ and a radius which increases without limit. For this sequence the function $\operatorname{cth}(\xi\sqrt{z})$ is uniformly bounded on all Γ_m. Since on Γ_m for $m \to \infty$ with respect to $\lambda \in [0, \lambda_0]$

$$\Phi_1(z, \lambda) = \frac{\operatorname{sh}(\xi\sqrt{z})}{\xi\sqrt{z}}[1 + o(1)], \quad \Phi_2(z, \lambda) = \frac{1}{\xi}\, z\sqrt{z}\operatorname{sh}(\xi\sqrt{z})[1 + o(1)]$$

are uniform, then for sufficiently large numbers m starting with some m_N the functions $\Phi_1(z, \lambda)$ and $\Phi_2(z, \lambda)$ do not become zero on Γ_m for any values of $\lambda \in [0, \lambda_0]$, i.e. the trajectories of roots of functions Φ_1 and Φ_2 in the plane z in the case of increase in λ in the indicated interval do not intersect Γ_m. Let us hold fixed any circumference Γ_p, $p \geqslant m_N$ and let us observe the motion of roots keeping in mind the fact that imaginary roots of each of the functions $\Phi_1(z, \lambda)$ and $\Phi_2(z, \lambda)$ in the plane z can occur only in the form of conjugate pairs, while Φ_1 and Φ_2 do not have common roots. Apparently, when λ increases in the interval $[0, \lambda_0]$ the roots Φ_1 and Φ_2 move in the plane z in such a manner that just as for $\lambda = 0$ all roots of Φ_1 inside Γ_p remain simple negative and separated by roots of Φ_2 while the number of imaginary roots of Φ_2 is equal to two or zero. From readily formulated energetic considerations it follows that the real part of imaginary roots of Φ_2 is negative. Under these conditions examination of possible alternatives of locations of roots of Φ_2 in the plane z (it is easy to show that they are all realized for some positive values of parameters) finally leads us to the following conclusion.

For positive $\varkappa$, ξ and λ the function $\varphi(z)$ has a denumerable set of simple poles ζ_{nm}, $0 > \zeta_1 > \zeta_2 > \ldots$. This set includes all poles of $\varphi(z)$. In each of the intervals (ζ_{n+1}, ζ_n), $n = 1, 2, \ldots$ the function $\varphi(z)$ varies from $+\infty$ to $-\infty$. In this connection in each interval $(\zeta_{n+1}, \zeta_{mn})$ with the exception of perhaps one (ζ_{j+1}, ζ_j), where $\varphi(z)$ has three zeros $k_{02} \leqslant k_{01} \leqslant k_j$, the function $\varphi(z)$ has one and only one root k_n, $\varphi'(k_n) < 0$, $n \neq j$. The roots of $\varphi(z)$ are all included in negative roots k_n, $n = 1, 2, \ldots$ (the value $n = j$ is taken into account) and two roots k_{01}, k_{02} which are outside the interval $(\zeta_1, 0)$ for $0 < \varkappa < \varkappa_0$, or on $(\zeta_1, 0)$ for $\varkappa \geqslant \varkappa_0$ (the quantity $\varkappa_0$ is determined by values of parameters $\xi > 0$, $\lambda > 0$). If $k_{01} \notin (\zeta_1, 0)$, $k_{02} \notin (\zeta_1, 0)$, the roots k_{01}, k_{02} are either imaginary $k_{01} = \bar{k}_{02} = -\alpha + \beta i$ $(\alpha > 0,\ \beta > 0)$, or real, $\zeta_{j+1} < k_{02} \leqslant k_{01} \leqslant k_j < \zeta_j$ (the value $j \geqslant 1$ depends on parameters).

4. Spectral distribution of the angular velocity of the sphere. On the basis of obtained information about the spectrum, all types of spectral distribution of angular velocity of the sphere are determined by contour integration with utilization of circumferences Γ_m (for $z \to \infty$ on all Γ_m by construction $\varphi(z) = z^2 + o(z^2)$)

$$\omega_0 t = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{-A_0 e^{zt}}{\varphi(z)}\, dz$$

1) For $k_{01} = k_{02} = -\alpha + \beta i$, $\alpha > 0$, $\beta > 0$

$$\omega_0(t) = a_0 e^{-\alpha t}\cos(\beta t + \vartheta_0) + \sum_{n=1}^{\infty} a_n e^{k_n t} \tag{4.1}$$

$$\underset{n\geqslant 1}{a_n} = -\frac{A_0}{\varphi'(k_n)} > 0, \qquad a_0 = -\frac{2A_0}{|\varphi'(k_{01})|}, \qquad \vartheta_0 = -\arg\varphi'(k_{01})$$

2) For $\zeta_{j+1} < k_{02} \leqslant k_{01} \leqslant k_j < \zeta_j$

$$\omega_0(t) = S_j + \Sigma_{(j)}\, a_n e^{k_n t} \tag{4.2}$$

In Eq. (4.2) the symbol $\Sigma_{(j)}$ designates summation over all values $n \geqslant 1$ with the exception of the value j. For $n \neq j$ the coefficient $a_n = -A_0/\varphi'(k_n) > 0$. The sum of residues of function $-A_0 e^{zt}/\varphi(z)$ with respect to k_j, k_{01}, k_{02} is designated by S_j.

If $\zeta_{j+1} < k_{02} < k_{01} < k_j < \zeta_j$, then

$$S_j = a_j^{(1)} e^{k_j t} + a_j^{(2)} e^{k_{01} t} + a_j^{(3)} e^{k_{02} t}$$

$$a_j^{(1)} = -A_0/\varphi'(k_j) > 0, \qquad a_j^{(2)} = -A_0/\varphi'(k_{01}) < 0, \qquad a_j^{(3)} = -A_0/\varphi'(k_{02}) > 0$$

If $\zeta_{j+1} < k_{02} = k_{01} < k_j < \zeta_j$, then

$$S_j = a_j^{(1)} e^{k_j t} + a_j^{(2)} t e^{k_{01} t} + a_j^{(3)} e^{k_{01} t}$$

$$a_j^{(1)} = -\frac{A_0}{\varphi'(k_j)} > 0, \qquad a_j^{(2)} = -\frac{2A_0}{\varphi''(k_{01})} < 0, \quad a_j^{(3)} = \frac{2A_0\varphi'''(k_{01})}{3[\varphi''(k_{01})]^2} = -\Sigma_{(j)}\, a_n - a_j^{(1)} < 0$$

If $\zeta_{j+1} < k_{02} < k_{01} = k_j < \zeta_j$, then

$$S_j = a_j^{(1)} t e^{k_j t} + a_j^{(2)} e^{k_j t} + a_j^{(3)} e^{k_{02} t}$$

$$a_j^{(1)} = -\frac{2A_0}{\varphi''(k_j)} > 0, \qquad a_j^{(2)} = \frac{2A_0\varphi'''(k_j)}{3[\varphi''(k_j)]^2} < 0, \qquad a_j^{(3)} = -\frac{A_0}{\varphi'(k_{02})} > 0$$

If $\zeta_{j+1} < k_{02} = k_{01} = k_j < \zeta_j$, then

$$S_j = a_j^{(1)} t^2 e^{k_j t} + a_j^{(2)} t e^{k_j t} + a_j^{(3)} e^{k_j t}$$

$$a_j^{(1)} = -\frac{3A_0}{\varphi'''(k_j)} > 0, \; a_j^{(2)} = \frac{3A_0\varphi^{IV}(k_j)}{2[\varphi'''(k_j)]^2}, \qquad a_j^{(3)} = \frac{3A_0\varphi^{V}(k_j)}{10[\varphi'''(k_j)]^2} - \frac{3A_0[\varphi^{IV}(k_j)]^2}{8[\varphi'''(k_j)]^3} < 0$$

3) For $\zeta_1 < k_{02} \leqslant k_{01} < 0$

$$\omega_0(t) = S_0 + \sum_{n=1}^{\infty} a_n e^{k_n t}, \qquad a_n = -\frac{A_0}{\varphi(k_n)} > 0 \qquad (n \geqslant 1) \tag{4.3}$$

If $\zeta_1 < k_{02} < k_{01} < 0$, then

$$S_0 = a_0^{(1)} e^{k_{01} t} + a_0^{(2)} e^{k_{02} t}$$

$$a_0^{(1)} = -\frac{A_0}{\varphi'(k_{01})} < 0, \qquad a_0^{(2)} = -\frac{A_0}{\varphi'(k_{02})} > 0, \qquad a_0^{(1)} = -a_0^{(2)} - \sum_{n=1}^{\infty} a_n$$

If $\zeta_1 < k_{02} = k_{01} < 0$, then $\quad S_0 = a_0^{(1)} t e^{k_{01} t} + a_0^{(2)} e^{k_{01} t}$

$$a_0^{(1)} = -\frac{2A_0}{\varphi''(k_{01})} < 0, \quad a_0^{(2)} = -\sum_{n=1}^{\infty} a_n = \frac{2A_0\varphi'''(k_{01})}{3[\varphi''(k_{01})]^2} < 0$$

For $n \to \infty$ it follows from elementary calculations that

$$k_n = -\frac{\pi^2}{\xi^2} n^2 - 2\left(\frac{\lambda^2}{1+\xi\lambda} + \frac{\varkappa}{\xi}\right)[1 + o(1)], \qquad \varphi'(k_n) = -\frac{\pi^4}{2\varkappa\xi^3} n^4 [1 + o(1)] \qquad (4.4)$$

Equations (4.4) show that for $t > 0$ the series (4.1)–(4.3) can be differentiated term by term an innumberable number of times, for $t = 0$ once. All derivatives with respect to $\omega_0(t)$ starting with the second tend to infinity as $t \to 0$. Separating the principal terms for $t \to \infty$ in series (4.1), (4.2) and (4.3) we establish the following. If roots k_{01} and k_{02} are outside of the interval $(\zeta_1, 0)$, the sphere passes through the equilibrium position odd and finite number of times, or an infinite number of times. If roots k_{01} and k_{02} belong to the interval $(\zeta_1, 0)$, then the sphere does not pass through the position of equilibrium but only approaches it monotonically and indefinitely with an angular velocity which does not become zero for $t > 0$.

We note that this derivation applies also to the analogous problems of small rotational or longitudinal oscillations of an elastically coupled rigid plane in a viscous fluid, bounded by stationary walls which are parallel to the oscillating plane [3], or of an infinite cylinder in a fluid bounded by a stationary coaxial cylindrical wall. The analysis for the infinite cylinder does not differ substantially from the one carried out in this paper and leads to the same basic conclusions.

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COMPARISON OF RESULTS OF AN ANALYSIS OF TRANSIENT WAVES IN SHELLS AND PLATES BY ELASTICITY THEORY AND APPROXIMATE THEORIES (*)

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Transient strain wave propagation in elastic shells and plates caused by an effect (application of loading, communication of displacements or velocities) which grows to a maximum or exerts influence in a time interval less than the time of strain wave traversal of a path equal to the characteristic dimension of the middle surface is considered on the

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